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REMARKS ON THE DERIVATION OF SEVERAL SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS FROM A GENERALIZATION OF THE EINSTEIN EQUATIONS

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Abstract

A generalization of the Einstein equations with the cosmological constant is considered for complex line elements. Several second order semilinear partial differential equations are derived from them as semilinear field equations in homogeneous and isotropic spaces. The non-relativistic limits of the field equations are also considered. The properties of spatial expansion and contraction are studied based on energy estimates of the field equations. Several dissipative and anti-dissipative properties are remarked.

1. Introduction

The solutions of the Einstein gravitational equations describe the spacetimes which are expanding or contracting. In this paper, we consider the derivation of several semilinear partial differential equations in those spacetimes, and we put some remarks on the fundamental effects of spatial variances for the energy estimates of the equations. We use complex line elements to give a unified derivation of the equations. There is a long history of the study of the Einstein equations in the complex coordinates, and the complex coordinates have played important roles in the study of the general relativity (see e.g. [35, 37, 48]). However, when we consider the semilinear terms which are power-type of fractional order in the field equations, we are not able to use complex derivatives since the semilinear terms are not holomorphic functions. Instead, we use the following space. For any natural number n and any fixed real numbers $\omega = (\omega^0, \cdots, \omega^n) \in (-\pi/2, \pi/2]^{1+n}$, we consider a $(1+n)$-dimensional space $\mathbb{M}^{1+n}$ defined by

$$\mathbb{M}^{1+n} := \{z \in \mathbb{C}^{1+n} \mid z^\alpha = x^\alpha e^{i\omega^\alpha},\ x^\alpha \in \mathbb{R},\ 0 \le \alpha \le n\},$$

where $\mathbb{C}$ denotes the set of complex numbers. We consider a generalization of the Einstein equations for non-Hermitian complex line elements of the form $g_{\alpha\beta}(z)dz^\alpha dz^\beta$, where $\{g_{\alpha\beta}\}_{0\le\alpha,\beta\le n}$ are complex-valued functions for $z = (z^0, \cdots, z^n) \in \mathbb{M}^{1+n}$. Under the cosmological principle, we give the solution of the generalized Einstein equations as

$$g_{\alpha\beta}dz^\alpha dz^\beta = -c^2(dz^0)^2 + a(z^0)^2 q^2 \left(1 + \frac{k^2 r^2}{4}\right)^{-2} \sum_{j=1}^{n} (dz^j)^2, \tag{1.1}$$

where $c > 0$ is the speed of light, $q(\neq 0), k \in \mathbb{C}$ are constants, $r^2 = \sum_{\alpha=1}^{n} (z^\alpha)^2$, and $a(\cdot)$

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is a complex-valued function which denotes the scale-function of the space (see (3.10), below). This solution is the well-known Friedmann-Lemaître-Robertson-Walker metric when $z = (t, x^1, \cdots, x^n) \in \mathbb{R}^{1+n}$, $q = 1$ and $k^2 = 0, \pm 1$, where k^2 denotes the curvature of the space. There is a large body of literature on the generalization of the Einstein equations for Hermitian line elements and general dimensions (see e.g. [10, 11, 18, 19, 28, 29]). In this paper, we consider the non-Hermitian line element on $\mathbb{M}^{1+n}$, and we derive several semilinear partial differential equations in a unified way as follows.

For any function f on $\mathbb{M}^{1+n}$, we define the derivative $\partial_\alpha f(z)$ for $z \in \mathbb{M}^{1+n}$ by

$$\partial_\alpha f(z) := \lim_{\substack{h\to 0\\ h\in\mathbb{R}\setminus\{0\}}} \frac{f(z^0, \cdots, z^{\alpha-1}, z^\alpha + he^{i\omega^\alpha}, z^{\alpha+1}, \cdots, z^n) - f(z)}{he^{i\omega^\alpha}}. \tag{1.2}$$

Since $z = (z^0, \cdots, z^n) \in \mathbb{M}^{1+n}$ is parametrized by $x = (x^0, \cdots, x^n) \in \mathbb{R}^{1+n}$ by the relation

$$z^\alpha = x^\alpha e^{i\omega^\alpha}, \tag{1.3}$$

if we put $f_*(x) := f(z)$ with (1.3), then we have $\partial_\alpha f(z) = e^{-i\omega^\alpha}\partial f_*(x)/\partial x^\alpha$. We say that f is a C^1-function if f is differentiable in the sense (1.2) and $\partial_\alpha f$ is continuous on $\mathbb{M}^{1+n}$. Let us consider the background spacetime $\mathbb{M}^{1+n}$ with the line element (1.1). We put $q = 1$ and $k = 0$ in (1.1). As the equation of motion of the massive scalar field described by a complex-valued function $\phi = \phi(z^0, \cdots, z^n)$ with the mass m and a potential $\lambda|\phi|^{p-1}\phi^2/(p+1)$ for $\lambda \in \mathbb{C}$ and $1 \le p < \infty$, we derive the second order differential equation

$$-\frac{1}{c^2}\left(\partial_0^2 + \frac{n\partial_0 a}{a}\partial_0 + \frac{m^2c^4}{\hbar^2}\right)\phi + \frac{1}{a^2}\Delta_z\phi + \lambda|\phi|^{p-1}\phi = 0 \tag{1.4}$$

for $z \in \mathbb{M}^{1+n}$ as the Euler-Lagrange equation of a Lagrange density (see (4.1), below), where $\hbar$ is the reduced Planck constant (the Dirac constant), $\partial_0 := \partial/\partial z^0$ and $\Delta_z := \sum_{j=1}^n \partial^2/(\partial z^j)^2$. We note that the term $|\phi|^{p-1}\phi$ is not holomorphic, so that, the equation (1.4) could not be defined on the whole space $\mathbb{C}^{1+n}$. We also show that the nonrelativistic limit of (1.4) yields the equation

$$\pm i\frac{2m}{\hbar}\partial_0 u + \frac{1}{a^2}\Delta_z u + \lambda|uw|^{p-1}u = 0 \tag{1.5}$$

for $z \in \mathbb{M}^{1+n}$ with a suitable transform from ϕ to $u = u(z^0, \cdots, z^n)$ (see (4.5), below), where $i := (-1)^{1/2}$ and w is a weight function defined by $w(z^0) := b_0(a(0)/a(z^0))^{n/2}$ for a constant $b_0 \in \mathbb{C}$.

By the transform (1.3), the equations (1.4) and (1.5) give typical second order partial differential equations. For example, let us consider the simplest case that the scale-function is given by a constant function $a(\cdot) = 1$. From (1.4) and (1.5), we obtain the semilinear Klein-Gordon equation

$$\partial_t^2\phi - c^2\Delta_x\phi + \frac{m^2c^4}{\hbar^2}\phi - c^2\lambda|\phi|^{p-1}\phi = 0, \tag{1.6}$$

the semilinear Schrödinger equation

$$\pm i\frac{2m}{\hbar}\partial_t u + \Delta_x u + \lambda|u|^{p-1}u = 0, \tag{1.7}$$

the semilinear elliptic equation

$$\partial_t^2\phi + c^2\Delta_x\phi + \frac{m^2c^4}{\hbar^2}\phi - c^2\lambda|\phi|^{p-1}\phi = 0, \tag{1.8}$$

and the semilinear parabolic equation

$$\frac{2m}{\hbar}\partial_t u - \Delta_x u - i\lambda|u|^{p-1}u = 0 \tag{1.9}$$

(see (5.2) and (5.3), below, for the case of general $a(\cdot)$), where we denote the Laplacian by $\Delta_x := \sum_{j=1}^n \partial^2/(\partial x^j)^2$. For the elliptic equation (1.8), the variable ct can be naturally regarded as one of spatial variables (see the argument on (5.6), below). The terms $\lambda|\phi|^{p-1}\phi$ and $\lambda|u|^{p-1}u$ are fundamental semilinear terms in nonlinear theory to describe the self-interaction of the solution. For the last parabolic equation (1.9), we note that the dimension of $\hbar/m$ in the SI units is $\mathsf{M}^2\mathsf{S}^{-1}$ (M: meter, S: second), which is equivalent to the dimension of the thermal diffusivity K_1 of the heat equation $\partial_t u - K_1\Delta_x u = 0$, and also to the dimension of the diffusion coefficient K_2 of the diffusion equation $\partial_t u - K_2\Delta_x u = 0$. We are able to replace Δ_x with $\sum_{j=1}^{\ell} \partial^2/(\partial x^j)^2 - \sum_{j=\ell+1}^n \partial^2/(\partial x^j)^2$ for any $1 \le \ell \le n-1$ when $n \ge 2$ in the above equations. For example, we obtain the semilinear ultrahyperbolic equation

$$\partial_t^2\phi - c^2\left(\sum_{j=1}^{\ell}\frac{\partial^2}{(\partial x^j)^2} - \sum_{j=\ell+1}^{n}\frac{\partial^2}{(\partial x^j)^2}\right)\phi + \frac{m^2c^4}{\hbar^2}\phi - c^2\lambda|\phi|^{p-1}\phi = 0 \tag{1.10}$$

instead of the Klein-Gordon equation (1.6) (see e.g. [25] for the ultrahyperbolic equation).

It is well-known that the Schrödinger equation (1.7) is derived from the Klein-Gordon equation (1.6) by the transform $\phi(t,x) = u(t,x)e^{\mp imc^2t/\hbar}$ and the nonrelativistic limit $c \to \infty$. We extend this fact to (1.4) and we obtain (1.5). There are other ways to derive the above equations by the formal manner. For example, the equation (1.9) is obtained from (1.7) by the Wick rotation ([48], the change from t to it) formally. However, this rotation is not valid for the nonrelativistic limit since the transform $\phi(t,x) = u(t,x)e^{\pm mc^2t/\hbar}$ does not preserve the gauge invariance in the semilinear term. The aim of this paper is to remark that the generalization of the Einstein equations on $\mathbb{M}^{1+n}$ (see (2.11), below) are useful to derive the above semilinear partial differential equations in a unified way. As far as the author knows, the derivation of the semilinear parabolic equation (1.9) (including the semilinear complex Ginzburg-Landau equation (5.8), below) based on the nonrelativistic limit from the field equation (1.4) is new.

The solutions of the Klein-Gordon equation (1.6) and the Schrödinger equation (1.7) have the properties of waves, while the solutions of the equation (1.9) as the diffusion equation have the properties of particles. On $\mathbb{M}^{1+n}$, these equations are naturally unified in the forms of (1.4) and (1.5) even for the semilinear equations. And the properties of the equations (1.4) and (1.5) are dependent on the choice of $\mathbb{M}^{1+n}$ in $\mathbb{C}^{1+n}$, namely, dependent on the choice of $\{\omega^\alpha\}_{\alpha=0}^n$. In this sense, we are able to regard (1.4) and (1.5) as the equations which describe the energy including waves and particles. This fact reminds us of the wave-particle duality in [12] and [14].

The equation (1.4) includes the semilinear wave equation and the semilinear Klein-Gordon equation in the curved spacetimes. There are increasing number of papers which consider those equations. As closely related results, the global solutions for small data for the Klein-Gordon equation have been shown in asymptotically de Sitter spacetime in [4]. The wave equation with quadratic nonlinear term in asymptotically de Sitter and Kerr-de Sitter spacetimes has been considered in [23]. The global solutions for semilinear wave equations have been shown on the manifold with the time slices being real hyperbolic spaces in [1, 2, 31, 32] (see also the references in Section 5).

The equation (1.5) includes the semilinear heat equation and the semilinear Schrödinger equation. There is a large literature on the Cauchy problem for (1.7) and (1.9) (see e.g. [7, 8, 9, 45, 52]). The properties of semilinear Schrödinger equations of the form $(i\partial_t + \Delta_g)u = |u|^{p-1}u$ have been studied on a certain compact and noncompact Riemannian manifold (M, g), where Δ_g is the Laplace-Beltrami operator on (M, g). In the hyperbolic space $\mathbb{H}^n$, the dispersive effect on Schrödinger equations was considered in [3], and the global existence of solutions with finite energy has been shown in [24]. In the de Sitter spacetime, a dissipative effect of the spatial expansion on Schrödinger equations was shown in [34].

This paper is organized as follows. In Section 2, we show a generalization of the Einstein equations on the space $\mathbb{M}^{1+n}$. Although the method is a slight modification of the classical argument for the Einstein equations, or the restriction of the Einstein equations on $\mathbb{C}^{1+n}$ to $\mathbb{M}^{1+n}$, we show the outline for the completeness of the paper since this part is essential to derive our field equations in Section 3.

In Section 3, we consider the spatial variance (expansion and contraction) described by the scale-function $a(\cdot)$, which satisfies the Einstein equations with the cosmological constant in homogeneous and isotropic spaces. The studies of roles of the cosmological constant and the spatial variance are important to describe the history of the universe, especially, the inflation and the accelerating expansion of the universe (see e.g. [21], [27], [36], [38], [42], [44]). One of the scale-functions (the second line in (3.10), below) follows from the equation of state (see (3.8) with $\sigma = -1$, below) when we regard the cosmological constant as "the dark energy" in cosmology. There are a lot of studies by physical and geometrical approaches on the dark energy (see e.g. [5, 47] for the cosmological constant, and [26, 41] for the modified gravity). Several dissipative and anti-dissipative properties by the spatial variances have been pointed out for the Klein-Gordon equation and the Schrödinger equation in de Sitter spacetime (see [33] and [34] and the references therein). We study more general equations (1.4) and (1.5) for spacetimes described by (1.1) with the flat spatial curvature $k = 0$ in this paper. Further detailed analysis on the Cauchy problems of (1.4) and (1.5) will appear in the forthcoming paper.

In Sections 4 and 5, we derive the above equations from (1.4) to (1.10).

In Section 6, we consider the energy estimates of the equations (1.4) and (1.5). We show that the spatial variance described by the scale-function $a(\cdot)$ gives dissipative and anti-dissipative properties for the energy estimates. Especially, we show that the spatial expansion yields the dissipative properties, while the spatial contraction yields the anti-dissipative properties in general.

We put two appendices as Section 7 and Section 8.

In Section 7, we give some remarks on Vilenkin's model (see [46] and also [22]) of the birth of the universe in $\mathbb{M}^{1+n}$. In Vilenkin's model, the purely imaginary time axis it for $t \in \mathbb{R}$ plays an important role to describe the birth of the universe from "nothing" through the tunnel effect. This fact is one of our motivations to regard the axes of the spacetime as the lines in the complex plane (see (1.3)).

In Section 8, we consider the geodesic curves defined by the complex line elements, and we show that the conservation law of the Hamiltonian is dependent on the scale-function in local coordinates, while it is independent of the scale-function in proper time.

REMARK 1.1. The complex coordinates have played an important role in the study of the general relativity (see e.g. [35, 37, 48]). Our motivation to connect the complex line element $g_{\alpha\beta}dz^\alpha dz^\beta$ to several partial differential equations is from the following elementary observation for the unified derivation of the rotational transform and the Lorentz transform. The Riemann metric $(cdx^0)^2+(dx^1)^2+\cdots+(dx^n)^2$ and the Lorentz metric $-(cdx^0)^2+(dx^1)^2+\cdots+(dx^n)^2$ for $x=(x^0,x^1,\cdots,x^n)\in\mathbb{R}^{1+n}$ are unified in a single form $(cdz^0)^2+(dz^1)^2+\cdots+(dz^n)^2$ for $z\in\mathbb{C}^{1+n}$ since $(z^0,z^1,\cdots,z^n)=(x^0,x^1,\cdots,x^n)$ gives the Riemann metric and $(z^0,z^1,\cdots,z^n)=(ix^0,x^1,\cdots,x^n)$ gives the Lorentz metric. Let us consider two coordinates $z=(z^0,z^1,\cdots,z^n)\in\mathbb{C}^{1+n}$ and $z_*=(z_*^0,z_*^1,\cdots,z_*^n)\in\mathbb{C}^{1+n}$ which satisfy the invariance of line elements

$$(cdz^0)^2+\sum_{j=1}^n(dz^j)^2=(cdz_*^0)^2+\sum_{j=1}^n(dz_*^j)^2.$$

Let us assume $z^j=z_*^j$ for $2\le j\le n$ for simplicity. For $\theta\in\mathbb{C}$, the transform

$$\begin{pmatrix}cz_*^0\\ z_*^1\end{pmatrix}=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}cz^0\\ z^1\end{pmatrix} \tag{1.11}$$

satisfies this invariance. For any fixed $-\pi/2<\omega\le\pi/2$, let us consider the lines $z^0=it$, $z^1=x^1e^{i\omega}$, $z_*^0=it_*$ and $z_*^1=x_*^1e^{i\omega}$ in the complex plane $\mathbb{C}$, where $t,t_*,x^1,x_*^1\in\mathbb{R}$. Then (1.11) is rewritten as

$$\begin{pmatrix}ct_*\\ x_*^1\end{pmatrix}=\begin{pmatrix}\cos\theta & ie^{i\omega}\sin\theta\\ ie^{-i\omega}\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}ct\\ x^1\end{pmatrix}.$$

So that, if $\omega=\pi/2$ and $\theta\in\mathbb{R}$, then we have the rotational transform

$$\begin{pmatrix}ct_*\\ x_*^1\end{pmatrix}=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}ct\\ x^1\end{pmatrix}.$$

If $\omega=0$ and $\tau:=i\theta\in\mathbb{R}$ (namely, $\theta\in i\mathbb{R}$), then we have the Lorentz transform

$$\begin{pmatrix}ct_*\\ x_*^1\end{pmatrix}=\begin{pmatrix}\cosh\tau & \sinh\tau\\ \sinh\tau & \cosh\tau\end{pmatrix}\begin{pmatrix}ct\\ x^1\end{pmatrix}.$$

Therefore, the complex line element naturally unifies the rotational transform and the Lorentz transform. This observation is useful when we replace Lorentzian problems with elliptic problems vice versa. We extend this observation to the partial differential equations focused on the semilinear terms in this paper.

2. A generalization of the Einstein equations

In this section, we show the generalization of the Einstein equations on $\mathbb{M}^{1+n}$ for the completeness of the paper (see e.g. [6] and [13] for the classical derivation of the Einstein equations for three spatial dimensions and real line elements). In the following, Greek letters $\alpha,\beta,\gamma,\cdots$ run from 0 to n, Latin letters $j,k,\ell,\cdots$ run from 1 to n. We use the Einstein rule for the sum of indices of tensors, for example, $T^\alpha{}_\alpha := \sum_{\alpha=0}^n T^\alpha{}_\alpha$ and $T^j{}_j := \sum_{j=1}^n T^j{}_j$.

For any function f on $\mathbb{M}^{1+n}$, we put $f_*(x) := f(z)$ with (1.3). We define the integral $\int_{\mathbb{M}^{1+n}} f(z)dz$ by

$$\int_{\mathbb{M}^{1+n}} f(z)dz := e^{i\sum_{\alpha=0}^n \omega^\alpha} \int_{\mathbb{R}^{1+n}} f_*(x)dx. \tag{2.1}$$

For any two points $A = (a^0e^{i\omega^0}, \cdots, a^ne^{i\omega^n})$, $B = (b^0e^{i\omega^0}, \cdots, b^ne^{i\omega^n}) \in \mathbb{M}^{1+n}$ with $\{a^\alpha, b^\alpha\}_{\alpha=0}^n \subset \mathbb{R}$, we put

$$\int_A^B f(z)dz := e^{i\sum_{\alpha=0}^n \omega^\alpha} \int_J f_*(x)dx, \tag{2.2}$$

where we have put $J := (a^0, b^0) \times \cdots \times (a^n, b^n)$.

We collect some fundamental results as follows.

Lemma 2.1. *For any C^1-functions f, g on $\mathbb{M}^{1+n}$, put $f_*(x) := f(z)$, $g_*(x) := g(z)$ with* (1.3). *Then the following results hold.*

(1) *For any two points $A, B \in \mathbb{M}^{1+n}$, the integration by parts*

$$\int_A^B f(z)\partial_\alpha g(z)dz = \frac{e^{i\sum_{\beta=0}^n \omega^\beta}}{e^{i\omega^\alpha}} \{(fg)(B) - (fg)(A)\} - \int_A^B \partial_\alpha f(z)g(z)dz \tag{2.3}$$

holds. Especially,

$$\int_{\mathbb{M}^{1+n}} f(z)\partial_\alpha g(z)dz = -\int_{\mathbb{M}^{1+n}} \partial_\alpha f(z)g(z)dz \tag{2.4}$$

holds if f or g has the compact support.

(2) *For any functions $h = (h^0, \cdots, h^n)$ on $\mathbb{M}^{1+n}$ and any holomorphic function F, the Euler-Lagrange equation of the action $I := \int_{\mathbb{M}^{1+n}} F(h(z), \partial h(z))dz$ is given by*

$$\frac{\partial F}{\partial h^\alpha} - \frac{\partial}{\partial z^\beta}\left(\frac{\partial F}{\partial\left(\partial_\beta h^\alpha\right)}\right) = 0. \tag{2.5}$$

Proof. (1) Let $A = (a^0e^{i\omega^0}, \cdots, a^ne^{i\omega^n})$, $B = (b^0e^{i\omega^0}, \cdots, b^ne^{i\omega^n})$ for real numbers $a^0, \cdots, a^n, b^0, \cdots, b^n$. Put $J := (a^0, b^0) \times \cdots \times (a^n, b^n)$. By the definition (2.2), we have

$$\int_A^B f(z)\partial_\alpha g(z)dz = \frac{e^{i\sum_{\beta=0}^n \omega^\beta}}{e^{i\omega^\alpha}} \int_J f_*(x)\frac{\partial g_*}{\partial x^\alpha}(x)dx.$$

By the integration by parts for f_* and g_*, we have

$$\int_J f_*(x)\frac{\partial g_*}{\partial x^\alpha}(x)dx = (f_*g_*)(B_*) - (f_*g_*)(A_*) - \int_J \frac{\partial f_*}{\partial x^\alpha}(x)g_*(x)dx,$$

where we have put $A_* := (a^0, \cdots, a^n)$ and $B_* := (b^0, \cdots, b^n)$. By $(f_*g_*)(B_*) = (fg)(B)$ and $(f_*g_*)(A_*) = (fg)(A)$, we obtain (2.3). When f or g has the compact support, (2.4) holds since the boundary terms are vanished.

(2) Let $\{\delta h^\alpha\}_{\alpha=0}^n$ be any smooth functions on $\mathbb{M}^{1+n}$ with compact supports. The variation δI is given by

$$\delta I = \int_{\mathbb{M}^{1+n}} \frac{d}{d\varepsilon} F\left(h + \varepsilon\delta h, \partial_\beta(h + \varepsilon\delta h)\right)\Big|_{\varepsilon=0} dz.$$

We have

$$\frac{d}{d\varepsilon}F\left(h+\varepsilon\delta h, \partial_\beta(h+\varepsilon\delta h)\right)=\frac{\partial F}{\partial h^\alpha}\delta h^\alpha+\frac{\partial F}{\partial\left(\partial_\beta h^\alpha\right)}\partial_\beta\delta h^\alpha.$$

Since we have

$$\int_{\mathbb{M}^{1+n}}\frac{\partial F}{\partial\left(\partial_\beta h^\alpha\right)}\partial_\beta\delta h^\alpha dz=-\int_{\mathbb{M}^{1+n}}\partial_\beta\left(\frac{\partial F}{\partial\left(\partial_\beta h^\alpha\right)}\right)\delta h^\alpha dz$$

by the result (1), we obtain

$$\delta I=\int_{\mathbb{M}^{1+n}}\left\{\frac{\partial F}{\partial h^\alpha}-\partial_\beta\left(\frac{\partial F}{\partial\left(\partial_\beta h^\alpha\right)}\right)\right\}\delta h^\alpha dz.$$

So that, we have (2.5) as the Euler-Lagrange equation. □

We consider a bilinear symmetric complex-valued functional $\langle\cdot,\cdot\rangle$ on the vector space spanned by the vectors $\{\partial_\alpha\}_{0\le\alpha\le n}$. We put $g_{\alpha\beta}(z):=\langle\partial_\alpha,\partial_\beta\rangle$. We denote by $(g_{\alpha\beta}(z))$ the $(1+n)\times(1+n)$-matrix whose components are given by $\{g_{\alpha\beta}(z)\}_{0\le\alpha,\beta\le n}$. Put $g(z):=\det(g_{\alpha\beta}(z))$. Let $(g^{\alpha\beta}(z))$ be the inverse matrix of $(g_{\alpha\beta}(z))$.

We consider a line element

$$-(cd\tau)^2=(d\ell)^2:=g_{\alpha\beta}(z)dz^\alpha dz^\beta, \tag{2.6}$$

where τ denotes the proper time and we take the square root of $(cd\tau)^2$ as $-\pi<\arg(cd\tau)\le\pi$. We define dz by

$$dz=dz^0\wedge\cdots\wedge dz^n:=\sum_\sigma \operatorname{sgn}(\sigma)\,dz^{\sigma(0)}\cdots dz^{\sigma(n)},$$

where σ denotes the permutation of $\{0,\cdots,n\}$. For the change of variables x to $y=(y^0,\cdots,y^n)\in\mathbb{R}^{1+n}$ by $y=y(x)$, we consider the complex variables $w=(w^0,\cdots,w^n)$ by $w^\alpha=e^{i\omega^\alpha}y^\alpha$. Then we have $\det(\partial z^\alpha/\partial w^\beta)\in\mathbb{R}$, $(-g(z))^{1/2}=|\det(\partial w^\alpha/\partial z^\beta)|(-g(w))^{1/2}$, and

$$(-g(w))^{1/2}dw=\left(\operatorname{sgn}\det\left(\frac{\partial w^\alpha}{\partial z^\beta}\right)\right)\cdot(-g(z))^{1/2}dz$$

by direct calculations, where $g(w)$ denotes the determinant of $\left(g_{\alpha\beta}(w)\right)$ with $g_{\alpha\beta}(w):=\langle\partial/\partial w^\alpha,\partial/\partial w^\beta\rangle$, and we take the square root of $-g$ as $-\pi<\arg(-g)\le\pi$.

By direct calculations, we have the fundamental results

$$\begin{aligned}g_{\alpha\beta}\partial_\gamma g^{\alpha\beta}&=-(\partial_\gamma g_{\alpha\beta})g^{\alpha\beta},\\ \partial_\gamma g^{\alpha\beta}&=-g^{\alpha\mu}(\partial_\gamma g_{\mu\nu})g^{\nu\beta},\\ \partial_\gamma g&=gg^{\alpha\beta}\partial_\gamma g_{\alpha\beta}.\end{aligned}$$

We call the function $T^{\alpha\beta\cdots}{}_{\gamma\delta\cdots}$ the tensor if it satisfies

$$T^{\alpha\beta\cdots}{}_{\gamma\delta\cdots}(w)=\frac{\partial w^\alpha}{\partial z^\varepsilon}\frac{\partial w^\beta}{\partial z^\zeta}\frac{\partial z^\eta}{\partial w^\gamma}\frac{\partial z^\theta}{\partial w^\delta}\cdots T^{\varepsilon\zeta\cdots}{}_{\eta\theta\cdots}(z)$$

for any change of variables z to w in $\mathbb{M}^{1+n}$. We call the tensor T^α the contravariant tensor, and the tensor T_α the covariant tensor.

For any contravariant tensor T^α, we denote its parallel displacement from z to $z+w$ by $\widetilde{T}^\alpha(z+w) := T^\alpha(z) - \Gamma^\alpha{}_{\beta\gamma}(z)T^\beta(z)w^\gamma$, where $\Gamma^\alpha{}_{\beta\gamma}(z)$ denotes the proportional constant at z. We assume the symmetry condition $\Gamma^\alpha{}_{\beta\gamma} = \Gamma^\alpha{}_{\gamma\beta}$, and

$$\left(g_{\alpha\beta}\widetilde{T}^\alpha\widetilde{T}^\beta\right)(z+w) = \left(g_{\alpha\beta}T^\alpha T^\beta\right)(z) + O\Big(\sum_{0\le\alpha\le n}(w^\alpha)^2\Big)$$

for any T^α and w^α. Then we have the Christoffel symbol

$$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2}g^{\alpha\delta}\left(\partial_\beta g_{\delta\gamma} + \partial_\gamma g_{\beta\delta} - \partial_\delta g_{\beta\gamma}\right). \tag{2.7}$$

We define the covariant derivative ∇_β for T^α by

$$\nabla_\beta T^\alpha(z) := \lim_{w^\beta\to 0}\frac{T^\alpha(z+\overrightarrow{w^\beta}) - \widetilde{T}^\alpha(z+\overrightarrow{w^\beta})}{w^\beta} = \partial_\beta T^\alpha(z) + \Gamma^\alpha{}_{\beta\gamma}(z)T^\gamma(z),$$

where $\overrightarrow{w^\beta} := (0,\cdots,0,w^\beta,0,\cdots,0)$ whose β-component is w^β and the other components are 0. In general, we define

$$\begin{aligned}\nabla_\delta T^{\alpha\beta\cdots}{}_{\mu\nu\cdots} := \partial_\delta T^{\alpha\beta\cdots}{}_{\mu\nu\cdots} + \Gamma^\alpha{}_{\delta\varepsilon}T^{\varepsilon\beta\cdots}{}_{\mu\nu\cdots} + \Gamma^\beta{}_{\delta\varepsilon}T^{\alpha\varepsilon\cdots}{}_{\mu\nu\cdots} + \cdots \\ - \Gamma^\varepsilon{}_{\delta\mu}T^{\alpha\beta\cdots}{}_{\varepsilon\nu\cdots} - \Gamma^\varepsilon{}_{\delta\nu}T^{\alpha\beta\cdots}{}_{\mu\varepsilon\cdots} - \cdots\end{aligned}$$

for any tensor $T^{\alpha\beta\cdots}{}_{\mu\nu\cdots}$. We note that $\nabla_\gamma g_{\alpha\beta} = 0$ and $\nabla_\gamma g^{\alpha\beta} = 0$ follow from (2.7). By direct calculations, we have

$$\Gamma^\beta{}_{\alpha\beta} = \partial_\alpha\left(\log(-g)^{1/2}\right), \tag{2.8}$$

$$\nabla_\alpha T^\alpha = \frac{1}{(-g)^{1/2}}\partial_\beta\left((-g)^{1/2}T^\beta\right), \tag{2.9}$$

$$\nabla_\alpha\nabla^\alpha\psi = \frac{1}{(-g)^{1/2}}\partial_\beta\left((-g)^{1/2}g^{\beta\gamma}\partial_\gamma\psi\right) \tag{2.10}$$

for any tensor T^α and any scalar ψ.

We define the Riemann curvature tensor

$$R^\delta{}_{\alpha\beta\gamma} := \partial_\beta\Gamma^\delta{}_{\alpha\gamma} - \partial_\gamma\Gamma^\delta{}_{\alpha\beta} + \Gamma^\delta{}_{\varepsilon\beta}\Gamma^\varepsilon{}_{\alpha\gamma} - \Gamma^\delta{}_{\varepsilon\gamma}\Gamma^\varepsilon{}_{\alpha\beta}$$

which is derived from $R^\delta{}_{\alpha\beta\gamma}T^\alpha = (\nabla_\beta\nabla_\gamma - \nabla_\gamma\nabla_\beta)T^\delta$. We define the Ricci tensor $R_{\alpha\beta} := R^\gamma{}_{\alpha\beta\gamma}$, and the scalar curvature $R := g^{\alpha\beta}R_{\alpha\beta}$. We define the Einstein tensor by $G_{\alpha\beta} := R_{\alpha\beta} - g_{\alpha\beta}R/2$. The change of upper and lower indices is done by $g_{\alpha\beta}$ and $g^{\alpha\beta}$, for example, $G^\alpha{}_\beta := g^{\alpha\gamma}G_{\gamma\beta}$.

Let $\Lambda \in \mathbb{C}$ be a constant, which is called the cosmological constant. Let us consider the variation by $g_{\alpha\beta}$ of the Einstein-Hilbert action $\int_{\mathbb{M}^{1+n}}(R+2\Lambda)(-g)^{1/2}\,dz$. When $\mathbb{M}^{1+n} = \mathbb{R}^{1+n}$, it is well-known that the Euler-Lagrange equation for the Einstein-Hilbert action is given by the Einstein equations $G_{\alpha\beta} - \Lambda g_{\alpha\beta} = 0$ in the vacuum. We have the following extension of this result for $\mathbb{M}^{1+n}$.

Proposition 2.2. *The Euler-Lagrange equation of* $\int_{\mathbb{M}^{1+n}}(R+2\Lambda)(-g)^{1/2}dz$ *is given by* $G_{\alpha\beta} - \Lambda g_{\alpha\beta} = 0$.

Proof. By the definitions of the Ricci tensor and the covariant derivative, and by the symmetry condition $\Gamma^{\sigma}{}_{\nu\mu} = \Gamma^{\sigma}{}_{\mu\nu}$, we have

$$\delta R_{\rho\mu} = \nabla_{\mu}\left(\delta\Gamma^{\lambda}{}_{\rho\lambda}\right) - \nabla_{\lambda}\left(\delta\Gamma^{\lambda}{}_{\rho\mu}\right),$$

where $\delta T^{\alpha\beta\cdots}{}_{\mu\nu\cdots}$ denotes the variation of $T^{\alpha\beta\cdots}{}_{\mu\nu\cdots}$ by $g_{\alpha\beta}$. Since we have $\delta R = (\delta g^{\alpha\beta})R_{\alpha\beta} + g^{\alpha\beta}\delta R_{\alpha\beta}$ and $g^{\alpha\beta}\delta R_{\alpha\beta} = \nabla_{\beta}A^{\beta}$, where we have put $A^{\beta} := g^{\alpha\beta}\delta\Gamma^{\lambda}{}_{\alpha\lambda} - g^{\alpha\lambda}\delta\Gamma^{\beta}{}_{\alpha\lambda}$, we have

$$\delta(R + 2\Lambda) = (\delta g^{\alpha\beta})R_{\alpha\beta} + \frac{1}{(-g)^{1/2}}\partial_{\gamma}\left((-g)^{1/2}A^{\gamma}\right)$$

by (2.9). Since we have $\delta(-g)^{1/2} = -(-g)^{1/2}g_{\alpha\beta}\left(\delta g^{\alpha\beta}\right)/2$, we obtain

$$\delta\int_{\mathbb{M}^{1+n}}(R + 2\Lambda)(-g)^{1/2}dz = \int_{\mathbb{M}^{1+n}}\left(G_{\alpha\beta} - \Lambda g_{\alpha\beta}\right)(-g)^{1/2}\delta g^{\alpha\beta}dz + \int_{\mathbb{M}^{1+n}}\partial_{\gamma}\left((-g)^{1/2}A^{\gamma}\right)dz.$$

Since the second term in the right hand side vanishes by the divergence theorem, the Euler-Lagrange equation is given by $G_{\alpha\beta} - \Lambda g_{\alpha\beta} = 0$. □

For a stress-energy tensor $T^{\alpha}{}_{\beta}$, we define the $(1 + n)$-dimensional Einstein equations

$$G^{\alpha}{}_{\beta} - \Lambda g^{\alpha}{}_{\beta} = \kappa\, T^{\alpha}{}_{\beta}, \tag{2.11}$$

where κ is a constant and we assume that κ is written as $\kappa = \kappa_0/c^4$ for some constant κ_0 which is independent of c. For the case $n = 3$ and real line elements, the constant κ is called the Einstein gravitational constant which is given by $\kappa = 8\pi\mathcal{G}/c^4$, where $\mathcal{G}$ is the Newton gravitational constant. For the case $n \geq 3$ and complex line elements, we are able to generalize the constant κ to

$$\kappa = \frac{2(n-1)\pi^{n/2}\mathcal{G}}{(n-2)\Gamma(n/2)c^4}, \tag{2.12}$$

where Γ denotes the gamma function (see Remark 2.3). We have obtained the generalized Einstein equations (2.11) with (2.12) for the complex line elements on $\mathbb{M}^{1+n}$.

Remark 2.3. Let us show the derivation of (2.12). We denote the volume of the unit ball in $\mathbb{R}^n$ by $\Omega_n := 2\pi^{n/2}/n\Gamma(n/2)$. We put $\hat{z} := (z^1, \cdots, z^n)$, $r(\hat{z}) := \left\{\sum_{j=1}^{n}(z^j)^2\right\}^{1/2}$, and $\omega^1 = \cdots = \omega^n$ in (1.3). We define a function $E(\hat{z})$ by

$$E(\hat{z}) := \begin{cases} \dfrac{1}{(2-n)n\Omega_n}r(\hat{z})^{2-n} & \text{if } n \geq 3, \\ \dfrac{1}{n\Omega_n}\log r(\hat{z}) & \text{if } n = 2, \\ \dfrac{1}{n\Omega_n}r(\hat{z}) & \text{if } n = 1. \end{cases}$$

Since $E(\hat{x})$ for $\hat{x} = (x^1, \cdots, x^n) \in \mathbb{R}^n$ is the fundamental solution of the Laplacian, the function $E(\hat{z})$ satisfies

$$\Delta_{\hat{z}}E(\hat{z}) = \delta(\hat{z}), \tag{2.13}$$

where $\Delta_{\hat{z}} := \sum_{j=1}^{n}\partial^2/(\partial z^j)^2$ and δ denotes the Dirac δ-function. We assume that $(g_{\alpha\beta})$ is sufficiently close to the Minkowski matrix $(\eta_{\alpha\beta}) := \mathrm{diag}(-c^2, 1, \cdots, 1)$. Namely, we put

$h_{\alpha\beta} := g_{\alpha\beta} - \eta_{\alpha\beta}$, and we assume that $|h_{\alpha\beta}|$ is sufficiently small. For a potential $\phi = \phi(\hat{z})$ and the Lagrangian $L(\hat{z}, d\hat{z}/d\tau) := \sum_{j=1}^{n}(dz^j/d\tau)^2/2 - \phi(\hat{z})$, the Euler-Lagrange equation for the action $\int L(\hat{z}, d\hat{z}/d\tau)d\tau$ is given by

$$\frac{d^2\hat{z}}{d\tau^2} + \nabla_{\hat{z}}\phi = 0, \tag{2.14}$$

where we have put $\nabla_{\hat{z}} := (\partial_1, \cdots, \partial_n)$. We regard this equation as the Newton equation of motion for the gravitational potential $\phi := n\Omega_n \mathcal{G}\rho *_{\hat{z}} E$, where $\rho = \rho(\hat{z})$ denotes the density of the mass. We note that

$$\Delta_{\hat{z}}\phi = n\Omega_n\mathcal{G}\rho \tag{2.15}$$

holds by (2.13). The Euler-Lagrange equation for the action

$$\int \left(-g_{\alpha\beta}\frac{dz^\alpha}{d\tau}\frac{dz^\beta}{d\tau}\right)^{1/2} d\tau$$

yields the equation of the geodesic curve as

$$\frac{d^2z^\delta}{d\tau^2} + \Gamma^\delta{}_{\alpha\beta}\frac{dz^\alpha}{d\tau}\frac{dz^\beta}{d\tau} = 0. \tag{2.16}$$

Let us consider the weak gravity $\partial_0 g_{\alpha\beta} \doteqdot 0$, $g^{0k}\partial_k g_{00} \doteqdot 0$, $h^{\alpha\beta}\partial_\gamma h_{\delta\varepsilon} \doteqdot 0$ and the slow particle $dz^j/dz^0 \doteqdot 0$. Then we have $\Gamma^\lambda{}_{00} \doteqdot -g^{\lambda k}\partial_k g_{00}/2$, $\Gamma^0{}_{00} \doteqdot 0$, $\Gamma^j{}_{00} \doteqdot -\partial_j g_{00}/2$ and $d\tau \doteqdot dz^0$ by the definitions of $\Gamma^\alpha{}_{\beta\gamma}$ and $d\tau$. So that, we have $d^2z^j/(dz^0)^2 + \Gamma^j{}_{00} \doteqdot 0$ which yields

$$\partial_j\left(\phi + \frac{g_{00}}{2}\right) \doteqdot 0 \tag{2.17}$$

for $1 \le j \le n$ by (2.14) and (2.16). Let us consider the case $\Lambda = 0$ in (2.11). We have

$$(n-1)R = -2\kappa T \tag{2.18}$$

and

$$(n-1)R_{\alpha\beta} = \kappa\Big((n-1)T_{\alpha\beta} - Tg_{\alpha\beta}\Big) \tag{2.19}$$

when $n \ge 1$ by (2.11), where n is an arbitrary natural number. Under the assumption $\partial_\alpha h_{\beta\gamma}\partial_\delta h_{\varepsilon\zeta} \doteqdot 0$, we have

$$R_{00} \doteqdot -\partial_j\Gamma^j{}_{00} \doteqdot \frac{1}{2}\Delta_{\hat{z}}g_{00} \doteqdot -\Delta_{\hat{z}}\phi = -n\Omega_n\mathcal{G}\rho, \tag{2.20}$$

where we have used the definition of Ricci tensor, the above fact $\Gamma^j{}_{00} \doteqdot -\partial_j g_{00}/2$, (2.17) and (2.15). We now consider the stress-energy tensor $T^{\alpha\beta}$ given by

$$T^{\alpha\beta} := -\rho\,\frac{\partial z^\alpha}{\partial\tau}\,\frac{\partial z^\beta}{\partial\tau}$$

based on the analogy to the stress tensor of the perfect gas. We have

$$T^{\alpha\beta} \doteqdot \begin{cases} -\rho & \text{if } (\alpha,\beta) = (0,0), \\ 0 & \text{if } (\alpha,\beta) \ne (0,0) \end{cases} \tag{2.21}$$

by $\partial z^j/\partial z^0 \doteqdot 0$. So that, we have

$$T \doteqdot -\rho g_{00} \tag{2.22}$$

and

$$T_{\alpha\beta} \doteqdot \begin{cases} -\rho(g_{00})^2 & \text{if } (\alpha,\beta) = (0,0), \\ 0 & \text{if } (\alpha,\beta) \neq (0,0). \end{cases} \tag{2.23}$$

Therefore, we obtain

$$n(n-1)\Omega_n \mathcal{G}\rho \doteqdot (n-2)\kappa\rho(g_{00})^2 \tag{2.24}$$

by (2.19) and (2.20). The required result (2.12) holds when $n \geq 3$ by $g_{00} \doteqdot -c^2$ and (2.24). When $n = 2$, we have $T^{\alpha\beta} \doteqdot 0$ since $\rho = 0$ by (2.24). When $n = 1$, we have $\kappa T^{\alpha\beta} \doteqdot 0$ since we have $\kappa = 0$ or $\rho = 0$ by (2.24).

3. Homogeneous and isotropic spaces

Let us derive the line element (1.1) as the solution of the Einstein equations (2.11). We put $r := \left\{\sum_{j=1}^{n}(z^j)^2\right\}^{1/2}$. We assume that the space is homogeneous and isotropic, and we consider the line element

$$g_{\alpha\beta}dz^\alpha dz^\beta := -c^2(dz^0)^2 + e^{h(z^0)}e^{f(r)}\sum_{j=1}^{n}(dz^j)^2, \tag{3.1}$$

where h and f are complex-valued functions. This line element is homogeneous in the sense that for any two points P and Q in $\mathbb{C}^n$, the ratio of the coefficients $e^{h(z^0)}e^{f(r(P))}/e^{h(z^0)}e^{f(r(Q))}$ is independent of z^0.

By direct calculations, we have $G^0{}_j = G^j{}_0 = 0$,

$$G^0{}_0 := \frac{n-1}{2c^2}\left\{\frac{n}{4}(\partial_0 h)^2 - c^2 e^{-h-f}\left(f'' + (n-1)\frac{f'}{r} + \frac{n-2}{4}(f')^2\right)\right\},$$

and

$$\begin{aligned} G^j{}_k := g^j{}_k &\left\{\frac{n-1}{2c^2}\left(\partial_0^2 h + \frac{n}{4}(\partial_0 h)^2\right)\right. \\ &\qquad \left. -\frac{n-2}{2}e^{-h-f}\left(f'' + (n-2)\frac{f'}{r} + \frac{n-3}{4}(f')^2\right)\right\} \\ &\qquad\qquad + \frac{n-2}{2}e^{-h-f}\left(f'' - \frac{f'}{r} - \frac{(f')^2}{2}\right)\frac{z^j z^k}{r^2}, \end{aligned}$$

where $f' := df/dr$. Since the space is isotropic, the coefficient of $z^j z^k$ must vanish. So that, f must satisfy $f'' - f'/r - (f')^2/2 = 0$, by which we obtain

$$e^{f(r)} = q^2\left(1 + \frac{k^2 r^2}{4}\right)^{-2} \tag{3.2}$$

for constants $q(\neq 0), k \in \mathbb{C}$. We define a function

$$a(z^0) := e^{h(z^0)/2}. \tag{3.3}$$

Let us consider the stress-energy tensor $T^\alpha{}_\beta$ of the perfect fluid

$$T^\alpha{}_\beta := \operatorname{diag}(\rho c^2, -p, \cdots, -p)$$

for constant density ρ and pressure p. We put $\widetilde{\rho} := \rho + \Lambda/\kappa c^2$ and $\widetilde{p} := p - \Lambda/\kappa$. Then (2.11) is rewritten as $G^\alpha{}_\beta = \kappa \cdot \operatorname{diag}(\widetilde{\rho} c^2, -\widetilde{p}, \cdots, -\widetilde{p})$. This equation shows that the cosmological constant $\Lambda > 0$ is regarded as the energy which has positive density and negative pressure in the vacuum $\rho = p = 0$ for $\kappa > 0$, by which we regard the cosmological constant Λ as "the dark energy." The equation $G^0{}_0 = \kappa\widetilde{\rho} c^2 g^0{}_0$ is rewritten as

$$\frac{n-1}{2}\left\{\left(\frac{\partial_0 a}{ca}\right)^2 + \frac{k^2}{q^2a^2}\right\} = \frac{\kappa c^2}{n}\cdot\widetilde{\rho}. \tag{3.4}$$

The equation $G^j{}_k = -\kappa\widetilde{p} g^j{}_k$ is rewritten as

$$\frac{n-1}{2}\left\{\frac{2}{n-2}\cdot\frac{\partial_0^2 a}{c^2 a} + \left(\frac{\partial_0 a}{ca}\right)^2 + \frac{k^2}{q^2a^2}\right\} = -\frac{\kappa}{n-2}\cdot\widetilde{p}, \tag{3.5}$$

which is rewritten as the Raychaudhuri equation

$$\frac{\partial_0^2 a}{c^2 a} = -\frac{n-2}{n-1}\cdot\kappa\left(\frac{\widetilde{\rho}c^2}{n} + \frac{\widetilde{p}}{n-2}\right) \tag{3.6}$$

by (3.4). Multiplying a^n to the both sides in (3.4), taking the derivative by z^0 variable, and using (3.5), we have the conservation of the mass

$$\partial_0(\widetilde{\rho}c^2 a^n) + \widetilde{p}\partial_0 a^n = 0. \tag{3.7}$$

For any number σ, we assume the equation of state

$$\widetilde{p} = \sigma\widetilde{\rho}c^2. \tag{3.8}$$

Then $a(z^0)$ must satisfy

$$\frac{\partial_0^2 a(z^0)}{c^2 a(z^0)} = -\frac{n-2+n\sigma}{n(n-1)}\cdot\kappa\widetilde{\rho}c^2$$

with

$$\widetilde{\rho} = \frac{n-1}{2}\cdot\frac{n}{\kappa c^4}\cdot\left(\frac{\partial_0 a(0)}{a(0)}\right)^2\cdot\left(\frac{a(0)}{a(z^0)}\right)^{n(1+\sigma)} \tag{3.9}$$

by (3.6) and (3.7), which has the solution

$$a(z^0) := \begin{cases} a(0)\left(1 + \frac{n(1+\sigma)\partial_0 a(0) z^0}{2a(0)}\right)^{2/n(1+\sigma)} & \text{if } \sigma \neq -1, \\ a(0)\exp\left(\frac{\partial_0 a(0) z^0}{a(0)}\right) & \text{if } \sigma = -1. \end{cases} \tag{3.10}$$

By the above argument, we have derived the line element

$$g_{\alpha\beta}dz^\alpha dz^\beta = -c^2(dz^0)^2 + a(z^0)^2 q^2\left(1 + \frac{k^2r^2}{4}\right)^{-2}\sum_{j=1}^n (dz^j)^2 \tag{3.11}$$

with (3.10) for constants $q(\neq 0), k \in \mathbb{C}$ as the solution of (2.11). This line element (3.11) is the required line element (1.1). We note that we have $k = 0$ under the assumption (3.8) by (3.4) and (3.9) since the scale-function a defined by (3.10) satisfies

$$\left(\frac{\partial_0 a(z^0)}{a(z^0)}\right)^2 = \left(\frac{\partial_0 a(0)}{a(0)}\right)^2 \cdot \left(\frac{a(0)}{a(z^0)}\right)^{n(1+\sigma)}.$$

Remark 3.1. The line element (3.11) is well-known as the Friedmann-Lemaître-Robertson-Walker metric for the case that $a(\cdot)(>0)$ is real-valued, $q = 1$ and $k^2 = 0, \pm 1$. Here, k^2 denotes the curvature of the space. In this case, $a(\cdot)$ in (3.10) blows up [resp. vanishes] in finite time when $\partial_0 a(0) > 0$ and $\sigma < -1$ [resp. $\partial_0 a(0) < 0$ and $\sigma > -1$], which is called Big-Rip [resp. Big-Crunch] in cosmology. The case $\sigma = -1$ shows the exponential expansion [resp. contraction] of $a(\cdot)$ when $\partial_0 a(0) > 0$ [resp. $\partial_0 a(0) < 0$]. The case $\sigma > -1$ [resp. $\sigma < -1$] shows the polynomial expansion [resp. contraction] of $a(\cdot)$ when $\partial_0 a(0) > 0$ [resp. $\partial_0 a(0) < 0$]. We draw the graph of $a(\cdot)$ in Figure 1, below, where $a_0 := a(0)$, $a_1 := \partial_0 a(0)$ and $T_0 := -2a(0)/n(1+\sigma)\partial_0 a(0)$. These models are studied for the expansion and the contraction of the universe. The line element (3.11) with (3.10) is a natural extension of these models for general dimensions and complex line elements.

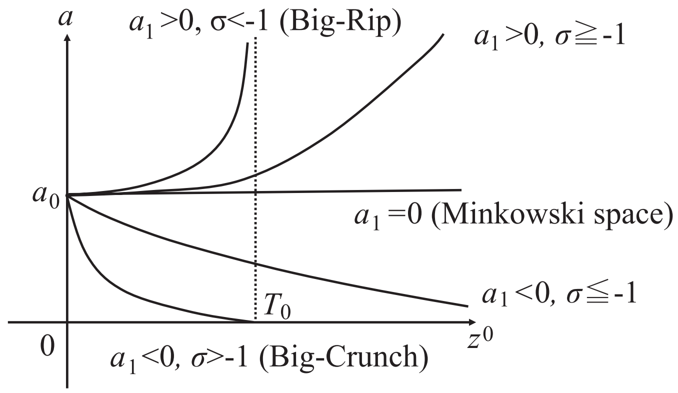

Fig. 1. Graph of the scale-function $a(\cdot)$ when $z^0 \in \mathbb{R}$

4. A field equation and the nonrelativistic limit

Let us derive the equations (1.4) and (1.5). For any $\lambda \in \mathbb{C}$ and any complex-valued C^2 function ϕ on $\mathbb{M}^{1+n}$, we define the Lagrangian

$$L(\phi) := -\frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \, \partial_\beta \phi - \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2 \phi^2 + \frac{\lambda}{p+1} |\phi|^{p-1} \phi^2. \tag{4.1}$$

Proposition 4.1. *For any $\phi \in C^2(\mathbb{M}^{1+n})$ and $\delta\phi \in C_0^2(\mathbb{M}^{1+n})$ with the constraint condition* $\arg \delta\phi = \arg \phi$*, the Euler-Lagrange equation for the action $\int_{\mathbb{M}^{1+n}} L(\phi)(-g)^{1/2} dz$ is given by*

$$\frac{1}{(-g)^{1/2}} \partial_\alpha ((-g)^{1/2} g^{\alpha\beta} \partial_\beta \phi) - \left(\frac{mc}{\hbar}\right)^2 \phi + \lambda |\phi|^{p-1} \phi = 0 \tag{4.2}$$

Proof. By the constraint condition, $\delta\phi$ is written as $\delta\phi = \eta e^{i \arg \phi}$ for nonnegative real-valued function η on $\mathbb{M}^{1+n}$. For $\varepsilon \in \mathbb{R}$, we put $\phi_\varepsilon := \phi + \varepsilon \delta\phi$. We have $\partial_\varepsilon \partial_\alpha \phi_\varepsilon = \partial_\alpha \delta\phi$ by $\partial_\alpha \phi_\varepsilon = \partial_\alpha \phi + \varepsilon \partial_\alpha \delta\phi$. Since we have

$$L(\phi_\varepsilon) = -\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi_\varepsilon\,\partial_\beta\phi_\varepsilon - \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2\phi_\varepsilon^2 + \frac{\lambda}{p+1}|\phi_\varepsilon|^{p-1}\phi_\varepsilon^2$$

and $|\phi_\varepsilon|^{p-1}\phi_\varepsilon^2 = (|\phi| + \varepsilon\eta)^{p+1}e^{2i\arg\phi}$, we have

$$\delta L(\phi) = \frac{d}{d\varepsilon}L(\phi_\varepsilon)\Big|_{\varepsilon=0} = -g^{\alpha\beta}\partial_\alpha\delta\phi\,\partial_\beta\phi - \left(\frac{mc}{\hbar}\right)^2\phi\delta\phi + \lambda|\phi|^p\eta e^{2i\arg\phi}.$$

Since we have $\lambda|\phi|^p\eta e^{2i\arg\phi} = \lambda|\phi|^{p-1}\phi\delta\phi$, we have

$$\begin{aligned}
&\delta\int_{\mathbb{M}^{1+n}} L(\phi)(-g)^{1/2}dz\\
=\ &\int_{\mathbb{M}^{1+n}} \delta L(\phi)(-g)^{1/2}dz\\
=\ &\int_{\mathbb{M}^{1+n}} \left\{\frac{1}{(-g)^{1/2}}\partial_\alpha((-g)^{1/2}g^{\alpha\beta}\partial_\beta\phi) - \left(\frac{mc}{\hbar}\right)^2\phi + \lambda|\phi|^{p-1}\phi\right\}\delta\phi(-g)^{1/2}dz,
\end{aligned}$$

where we have used the divergence theorem. So that, the Euler-Lagrange equation is given by (4.2) as required. □

We put $q = 1$ and $k = 0$ in (3.11). Since the line element (3.11) is rewritten as $-c^2(dz^0)^2 + a(z^0)^2\sum_{1\le j\le n}(dz^j)^2$, the field equation (4.2) is rewritten as

$$-\frac{1}{c^2}\left(\partial_0^2 + \frac{n\partial_0 a}{a}\partial_0 + \frac{m^2c^4}{\hbar^2}\right)\phi + \frac{1}{a^2}\Delta_z\phi + \lambda|\phi|^{p-1}\phi = 0, \tag{4.3}$$

which is the desired equation (1.4). For any constant $b_0 \in \mathbb{C}$, we define a weight function $w(z^0)$ and a function $b(z^0)$ by

$$w(z^0) := b_0\left(\frac{a(0)}{a(z^0)}\right)^{n/2}, \quad b(z^0) := w(z^0)\exp\left(\mp i\frac{mc^2}{\hbar}z^0\right), \tag{4.4}$$

where we note $b(0) = b_0$. We transform ϕ to u by the relation

$$\phi(z^0, \cdots, z^n) = u(z^0, \cdots, z^n)b(z^0). \tag{4.5}$$

We assume $mz^0/\hbar \in \mathbb{R}$. Putting (4.5) into (4.3), we have

$$\left(\partial_0^2 + \frac{n\partial_0 a}{a}\partial_0 + \frac{m^2c^4}{\hbar^2}\right)\phi = b\cdot\left(\partial_0^2 u + 2\partial_0 uA + uB\right),$$

where we have put

$$A := \frac{\partial_0 b}{b} + \frac{n\partial_0 a}{2a}, \quad B := \frac{\partial_0^2 b}{b} + \frac{n\partial_0 a}{a}\cdot\frac{\partial_0 b}{b} + \frac{m^2c^4}{\hbar^2}.$$

By the definition of $b(\cdot)$, we have

$$A = \mp i\frac{mc^2}{\hbar}, \quad B = -\left(\frac{n\partial_0 a}{2a}\right)^2 - \frac{n}{2}\partial_0\left(\frac{\partial_0 a}{a}\right).$$

We also have

$$\frac{1}{a^2}\Delta_z\phi + \lambda|\phi|^{p-1}\phi = b\cdot\left(\frac{1}{a^2}\Delta_z u + \lambda|uw|^{p-1}u\right),$$

where we have used the gauge invariance of $\lambda|\phi|^{p-1}\phi$ and the assumption $mz^0/\hbar \in \mathbb{R}$. So that, (4.3) is rewritten as

$$-\frac{1}{c^2}\left(\partial_0^2 u \mp i\,\frac{2mc^2}{\hbar}\partial_0 u + uB\right) + \frac{1}{a^2}\Delta_z u + \lambda|uw|^{p-1}u = 0.$$

The nonrelativistic limit ($c \to \infty$) of this equation yields

$$\pm i\,\frac{2m}{\hbar}\partial_0 u + \frac{1}{a^2}\Delta_z u + \lambda|uw|^{p-1}u = 0, \tag{4.6}$$

which is the desired equation (1.5).

5. A unified derivation of several PDEs

Let us derive the partial differential equations from (1.6) to (1.10). Let us consider the equations (4.3) and (4.6) under the transform (1.3) with $\omega^1 = \cdots = \omega^n$. We put $t = x^0$ and

$$\begin{aligned} \phi_*(t, x^1, \cdots, x^n) &:= \phi(z^0, z^1, \cdots, z^n), \\ u_*(t, x^1, \cdots, x^n) &:= u(z^0, z^1, \cdots, z^n), \\ a_*(t) := a(z^0), &\quad w_*(t) := w(z^0). \end{aligned} \tag{5.1}$$

We put $\theta := \arg a_*$. Then (4.3) is rewritten as

$$-\frac{1}{c^2}\frac{e^{2i(\theta+\omega^1)}}{e^{2i\omega^0}}\left(\partial_t^2 + \frac{n\partial_t a_*}{a_*}\partial_t + \left(\frac{mc^2 e^{i\omega^0}}{\hbar}\right)^2\right)\phi_* + \frac{1}{|a_*|^2}\Delta_x\phi_* + e^{2i(\theta+\omega^1)}\lambda|\phi_*|^{p-1}\phi_* = 0, \tag{5.2}$$

and (4.6) is rewritten as

$$\pm i\frac{2me^{i\omega^0}}{\hbar}\partial_t u_* + \frac{e^{2i\omega^0}}{e^{2i(\theta+\omega^1)}}\left(\frac{1}{|a_*|^2}\Delta_x u_* + e^{2i(\theta+\omega^1)}\lambda|u_*w_*|^{p-1}u_*\right) = 0. \tag{5.3}$$

The equation (5.2) and its nonrelativistic limit (5.3) give a unified derivation of the elliptic equation, the Klein-Gordon equation, the Schrödinger equation, and the parabolic equation as follows. For simplicity, we consider the simplest case $a(\cdot) = 1$ which follows from $a(0) = 1$ and $\partial_0 a(0) = 0$ in (3.10). So that, we have $\theta = 0$ and $a_* = 1$. We put $b_0 = 1$ in (4.4), which yields $w_* = 1$. When $\omega^0 = \cdots = \omega^n = 0$, (5.2) and (5.3) are rewritten as the Klein-Gordon equation

$$\partial_t^2\phi_* + \frac{m^2c^4}{\hbar^2}\phi_* - c^2\Delta_x\phi_* - c^2\lambda|\phi_*|^{p-1}\phi_* = 0, \tag{5.4}$$

and the Schrödinger equation

$$\pm i\frac{2m}{\hbar}\partial_t u_* + \Delta_x u_* + \lambda|u_*|^{p-1}u_* = 0, \tag{5.5}$$

respectively. When $\omega^0 = 0$ and $\omega^1 = \cdots = \omega^n = \pi/2$, (5.2) is rewritten as the elliptic equation

$$\partial_t^2\phi_* + \frac{m^2c^4}{\hbar^2}\phi_* + c^2\Delta_x\phi_* - c^2\lambda|\phi_*|^{p-1}\phi_* = 0, \tag{5.6}$$

where we should note that the line element (3.11) with $q = 1$ and $k = 0$ becomes

$$g_{\alpha\beta}dz^\alpha dz^\beta = -c^2(dt)^2 - \sum_{j=1}^{n}(dx^j)^2,$$

by which the variable ct is naturally regarded as one of the spatial variables (namely, there is no difference between cdt and $\{dx^j\}_{j=1}^n$). When $\omega^0 = 0$ and $\omega^1 = \cdots = \omega^n = \pi/4$, (5.3) with the positive sign is rewritten as the parabolic equation

$$\frac{2m}{\hbar}\partial_t u_* - \Delta_x u_* - i\lambda|u_*|^{p-1}u_* = 0. \tag{5.7}$$

So that, we have obtained the equations (1.6), (1.7), (1.8) and (1.9). We obtain the equation (1.10) when $\omega^0 = \cdots = \omega^\ell = 0$ and $\omega^{\ell+1} = \cdots = \omega^n = \pi/2$.

We are also able to derive the complex Ginzburg-Landau equation

$$\partial_t u_* - \gamma\Delta u_* - \lambda_1 u_* + \lambda_2|u_*|^2 u_* = 0, \tag{5.8}$$

where $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > 0$, $\lambda_1 \geq 0$, $\lambda_2 \in \mathbb{C}$ with $\operatorname{Re}\lambda_2 > 0$, as the sum of the potentials with $p = 1$ and $p = 3$ in (5.3) when $\gamma = \pm i\hbar/2me^{2i\omega^1}$, $\mp\hbar(\sin 2\omega^1)/2m > 0$, $\theta = \omega^0 = 0$ and $a_* = 1$. We refer to [20], [30], [39, (2.1)], [40], [43, (II.1) and the footnote in p.304] for the complex Ginzburg-Landau equation.

We note that (5.2) yields the semilinear Klein-Gordon equation

$$\partial_t^2\phi_* + nH\partial_t\phi_* + \frac{m^2c^4}{\hbar^2}\phi_* - \frac{c^2}{e^{2Ht}}\Delta_x\phi_* - c^2\lambda|\phi_*|^{p-1}\phi_* = 0 \tag{5.9}$$

in the de Sitter spacetime when $\omega^0 = \cdots = \omega^n = 0$ and $a_*(t) = e^{Ht}$ with the Hubble constant $H \in \mathbb{R}$. The Klein-Gordon equation in de Sitter spacetime has been considered in [15] for the tachyonic field, in [51] for the Huygens' principle, and in [50] for the Higgs scalar field. The Cauchy problem of the semilinear Klein-Gordon equations has been considered in [17, 33, 49] in the de Sitter spacetime, and in [16] in the Friedmann-Lemaître-Robertson-Walker spacetime. From the term $nH\partial_t\phi_*$ in (5.9), we know that the spatial expansion $H > 0$ yields the effect of dissipation, while the spatial contraction $H < 0$ yields the effect of anti-dissipation. In [34], we have considered the Cauchy problem of the semilinear Schrödinger equation in the de Sitter spacetime which is derived from (5.9). Our equations (5.2) and (5.3) involve the equations in [33] and [34] as a part. The Cauchy problem of (5.2) and (5.3) will be studied in the forthcoming paper.

6. Energy estimates

We have derived the equations (4.3) and (4.6). The properties of the equations depend on the local coordinate (t, x) defined by $t := x^0$ and (1.3). In this section, we show that the equations have dissipative and anti-dissipative properties on energy estimates dependently on $\omega^0, \cdots, \omega^n$ in (1.3).

We assume that there exist two functions V_0 and V_0' on $\mathbb{C}$ which satisfy

$$\partial_t V_0(\psi) = \operatorname{Re}\left(\partial_t\overline{\psi}\, V_0'(\psi)\right) \tag{6.1}$$

for any complex-valued function $\psi = \psi(t, x)$. We put $\omega^1 = \cdots = \omega^n$, $t = x^0$. We assume that $\theta := \arg a_*$ is a constant. We put $V' := -e^{-2i(\theta+\omega^1)}V_0'$. Let us consider energy estimates for the equations

$$-\frac{1}{c^2}\left(\partial_0^2+\frac{n\partial_0 a}{a}\partial_0+\frac{m^2c^4}{\hbar^2}\right)\phi+\frac{1}{a^2}\Delta_z\phi+V'(\phi)=0 \tag{6.2}$$

and

$$\pm i\frac{2m}{\hbar}\partial_0 u+\frac{1}{a^2}\Delta_z u+\frac{1}{w}V'(uw)=0, \tag{6.3}$$

which are extensions of (4.3) and (4.6) for general nonlinear terms V'. When $V'(\phi)=\lambda|\phi|^{p-1}\phi$, we have (4.3) and (4.6). For example,

$$V_0(\psi):=\frac{\lambda_0|\psi|^{p+1}}{p+1},\quad V_0'(\psi):=\lambda_0|\psi|^{p-1}\psi,\quad \lambda_0\in\mathbb{R} \tag{6.4}$$

satisfies (6.1). We use (5.1). The equations (6.2) and (6.3) are rewritten as

$$-\frac{1}{c^2}\frac{e^{2i(\theta+\omega^1)}}{e^{2i\omega^0}}\left(\partial_t^2+\frac{n\partial_t a_*}{a_*}\partial_t+\left(\frac{mc^2e^{i\omega^0}}{\hbar}\right)^2\right)\phi_*+\frac{1}{|a_*|^2}\Delta_x\phi_*-V_0'(\phi_*)=0, \tag{6.5}$$

and

$$\pm i\frac{2me^{i\omega^0}}{\hbar}\partial_t u_*+\frac{e^{2i\omega^0}}{e^{2i(\theta+\omega^1)}}\left(\frac{1}{|a_*|^2}\Delta_x u_*-\frac{1}{w_*}V_0'(u_*w_*)\right)=0. \tag{6.6}$$

We put $C_0:=2me^{i\omega^0}/\hbar$. We have the energy estimate for (6.5) as follows.

Proposition 6.1 (Energy estimates)**.** *Let us consider* (6.5). *Assume* $C_0\in\mathbb{R}$, $e^{2i(\theta+\omega^1)}/e^{2i\omega^0}\in\mathbb{R}$ *and* (6.1). *Then we have*

$$\int_{\mathbb{R}^n}e^0(t,x)dx+\int_0^t\int_{\mathbb{R}^n}e^{n+1}(s,x)dxds=\int_{\mathbb{R}^n}e^0(0,x)dx, \tag{6.7}$$

where we have put

$$e^0:=\frac{e^{2i(\theta+\omega^1)}}{c^2e^{2i\omega^0}}\left(|\partial_t\phi_*|^2+\frac{C_0^2c^4}{4}|\phi_*|^2\right)+\frac{1}{|a_*|^2}\sum_{j=1}^n|\partial_{x^j}\phi_*|^2+2V_0(\phi_*)$$

and

$$e^{n+1}:=\frac{e^{2i(\theta+\omega^1)}}{c^2e^{2i\omega^0}}2\mathrm{Re}\left(\frac{n\partial_t a_*}{a_*}\right)|\partial_t\phi_*|^2-\partial_t\left(\frac{1}{|a_*|^2}\right)\sum_{j=1}^n|\partial_{x^j}\phi_*|^2.$$

Proof. We put

$$e^j:=-\frac{2}{|a_*|^2}\mathrm{Re}\left(\partial_t\overline{\phi_*}\partial_{x^j}\phi_*\right)$$

for $1\le j\le n$. Multiplying $\partial_t\overline{\phi_*}$ to the both sides in (6.5) and taking their real parts, we have

$$\begin{aligned}&-\frac{1}{c^2}\frac{e^{2i(\theta+\omega^1)}}{e^{2i\omega^0}}\left(\partial_t|\partial_t\phi_*|^2 2\mathrm{Re}\left(\frac{n\partial_t a_*}{a_*}\right)|\partial_t\phi_*|^2+\left(\frac{mc^2e^{i\omega^0}}{\hbar}\right)^2\partial_t|\phi_*|^2\right)\\&\qquad+\frac{1}{|a_*|^2}\sum_{j=1}^n\left\{\partial_{x^j}2\mathrm{Re}\left(\partial_t\overline{\phi_*}\partial_{x^j}\phi_*\right)-\partial_t|\partial_{x^j}\phi_*|^2\right\}-2\partial_t V_0(\phi_*)=0,\end{aligned}$$

where we have used $C_0\in\mathbb{R}$, $e^{2i(\theta+\omega^1)}/e^{2i\omega^0}\in\mathbb{R}$, (6.1),

$$2\mathrm{Re}(\partial_t\overline{\phi_*}\partial_t^2\phi_*) = \partial_t|\partial_t\phi_*|^2, \quad 2\mathrm{Re}(\partial_t\overline{\phi_*}\phi_*) = \partial_t|\phi_*|^2$$

and

$$\mathrm{Re}\left(\partial_t\overline{\phi_*}V_0'(\phi_*)\right) = \partial_t V_0(\phi_*).$$

Since we have

$$\frac{1}{|a_*|^2}\sum_{j=1}^{n}\left\{\partial_{x^j}2\mathrm{Re}\left(\partial_t\overline{\phi_*}\partial_{x^j}\phi_*\right) - \partial_t|\partial_{x^j}\phi_*|^2\right\}$$

$$= -\sum_{j=1}^{n}\partial_{x^j}e^j - \partial_t\left(\frac{1}{|a_*|^2}\sum_{j=1}^{n}|\partial_{x^j}\phi_*|^2\right) + \left(\partial_t\frac{1}{|a_*|^2}\right)\sum_{j=1}^{n}|\partial_{x^j}\phi_*|^2,$$

we obtain

$$\partial_t e^0 + \sum_{j=1}^{n}\partial_{x^j}e^j + e^{n+1} = 0.$$

The required result follows from the integration for t and x. □

Let us consider the case $\theta = \omega^0 = \omega^1 = 0$ in Proposition 6.1. Since we have $e^{2i(\theta+\omega^1)}/e^{2i\omega^0} = 1$, the equation (6.5) becomes the nonlinear Klein-Gordon equation

$$\partial_t^2\phi_* + \frac{n\partial_t a_*}{a_*}\partial_t\phi_* + \left(\frac{mc^2}{\hbar}\right)^2\phi_* - \frac{c^2}{|a_*|^2}\Delta_x\phi_* + c^2V_0'(\phi_*) = 0.$$

When the space is invariant, namely, when $a_*(\cdot)$ is a constant, we have $e^{n+1} = 0$. Thus, the strict energy conservation

$$\int_{\mathbb{R}^n} e^0(t,x)dx = \int_{\mathbb{R}^n} e^0(0,x)dx$$

holds by (6.7). When the space is expanding, namely, when $\partial_t a_*(\cdot) > 0$, we have $e^{n+1} > 0$, which yields the dissipative property on the energy estimate (6.7). Contrarily, when the space is contracting, namely, when $\partial_t a_*(\cdot) < 0$, we have $e^{n+1} < 0$, which yields the anti-dissipative property.

Next, we consider charge and energy estimates for the equation (6.6) as follows.

Proposition 6.2. *Let us consider* (6.6). *Assume* $C_0 \in \mathbb{R}$. *Let* V_0' *satisfy*

(6.8) $$\mathrm{Im}\{\bar{z}V_0'(z)\} = 0$$

for any $z \in \mathbb{C}$ *and its complex conjugate* $\bar{z}$.

(1) *(Charge estimates.) We have*

(6.9) $$\pm\int_{\mathbb{R}^n} e_C^0(t,x)dx + \int_0^t\int_{\mathbb{R}^n} e_C^{n+1}(s,x)dxds = \pm\int_{\mathbb{R}^n} e_C^0(0,x)dx,$$

where we have put

$$e_C^0 := C_0|u_*|^2$$

and

$$e_C^{n+1} := 2\mathrm{Im}\left(\frac{e^{2i(\theta+\omega^1)}}{e^{2i\omega^0}}\right)\left(\frac{1}{|a_*|^2}\sum_{j=1}^n |\partial_{x^j}u_*|^2 + \frac{1}{|w_*|^2}\overline{u_*w_*}V_0'(u_*w_*)\right).$$

(2) *(Energy estimates.) Assume* (6.1). *Then we have*

$$\int_{\mathbb{R}^n} e_E^0(t,x)dx + \int_0^t\int_{\mathbb{R}^n} e_E^{n+1}(s,x)dxds = \int_{\mathbb{R}^n} e_E^0(0,x)dx, \tag{6.10}$$

where we have put

$$e_E^0 := \sum_{j=1}^n |\partial_{x^j}u_*|^2 + \frac{2|a_*|^2}{|w_*|^2}V_0(u_*w_*)$$

and

$$e_E^{n+1} := \pm 2C_0\mathrm{Im}\left(\frac{e^{2i(\theta+\omega^1)}}{e^{2i\omega^0}}\right)|a_*|^2|\partial_t u_*|^2 + \frac{n}{2|w_*|^2}\left(\partial_t|a_*|^2\right)\cdot\left(\overline{u_*w_*}V_0'(u_*w_*) - \frac{2(n+2)}{n}V_0(u_*w_*)\right).$$

Proof. (1) We put

$$e_C^j := 2\mathrm{Im}\left(\frac{e^{2i\omega^0}}{|a_*|^2e^{2i(\theta+\omega^1)}}\overline{u_*}\partial_{x^j}u_*\right)$$

for $1 \le j \le n$. Multiplying $\overline{u_*}$ to the both sides in (6.6) and taking their imaginary parts, we have

$$\pm 2\mathrm{Im}\,(iC_0\overline{u_*}\partial_t u_*) + 2\mathrm{Im}\left(\frac{1}{|a_*|^2}\frac{e^{2i\omega^0}}{e^{2i(\theta+\omega^1)}}\sum_{j=1}^n\left\{\partial_{x^j}\left(\overline{u_*}\partial_{x^j}u_*\right) - |\partial_{x^j}u_*|^2\right\}\right) - \frac{1}{|w_*|^2}\left(2\mathrm{Im}\frac{e^{2i\omega^0}}{e^{2i(\theta+\omega^1)}}\right)\overline{u_*w_*}V_0'(u_*w_*) = 0,$$

where we have used (6.8). Since we have $C_0 \in \mathbb{R}$,

$$2\mathrm{Im}\,(iC_0\overline{u_*}\partial_t u_*) = C_0\partial_t|u_*|^2$$

and

$$2\mathrm{Im}\left(\frac{1}{|a_*|^2}\frac{e^{2i\omega^0}}{e^{2i(\theta+\omega^1)}}\sum_{j=1}^n\left\{\partial_{x^j}\left(\overline{u_*}\partial_{x^j}u_*\right) - |\partial_{x^j}u_*|^2\right\}\right) = \sum_{j=1}^n \partial_{x^j}e^j - \frac{1}{|a_*|^2}\left(2\mathrm{Im}\frac{e^{2i\omega^0}}{e^{2i(\theta+\omega^1)}}\right)\sum_{j=1}^n|\partial_{x^j}u_*|^2,$$

we obtain

$$\pm\partial_t e_C^0 + \sum_{j=1}^n \partial_{x^j}e_C^j + e_C^{n+1} = 0.$$

The required result follows from the integration for t and x.

(2) We rewrite (6.6) as

$$\pm iC_0 \frac{e^{2i(\theta+\omega^1)}}{e^{2i\omega^0}} |a_*|^2 \partial_t u_* + \sum_{j=1}^n \partial_{x^j}^2 u_* - \frac{|a_*|^2}{w_*} V_0'(u_* w_*) = 0.$$

Multiplying $\partial_t \overline{u_*}$ to the both sides in this equation and taking their real parts, we have

$$\mp C_0 2\mathrm{Im}\, \frac{e^{2i(\theta+\omega^1)}}{e^{2i\omega_0}} |a_*|^2 |\partial_t u_*|^2 - \sum_{j=1}^n \partial_{x^j} e_E^j - \partial_t \sum_{j=1}^n |\partial_{x^j} u_*|^2 - 2\mathrm{Re}\, I = 0, \tag{6.11}$$

where we have put $e_E^j := -2\mathrm{Re}\,(\partial_t \overline{u_*} \partial_{x^j} u_*)$ for $1 \le j \le n$ and

$$I := \frac{|a_*|^2}{w_*} \partial_t \overline{u_*} V_0'(u_* w_*).$$

Since we have

$$I = \frac{|a_*|^2}{|w_*|^2} \left(\partial_t (\overline{u_* w_*}) V_0'(u_* w_*) - \frac{\partial_t \overline{w_*}}{\overline{w_*}} \overline{u_* w_*} V_0'(u_* w_*) \right)$$

by

$$\partial_t \overline{u_*} = \frac{1}{\overline{w_*}} \left(\partial_t (\overline{u_* w_*}) - \frac{\partial_t \overline{w_*}}{\overline{w_*}} \overline{u_* w_*} \right),$$

we obtain

$$2\mathrm{Re}\, I = \frac{|a_*|^2}{|w_*|^2} \left(2\partial_t V_0(u_* w_*) - \left(2\mathrm{Re}\, \frac{\partial_t \overline{w_*}}{\overline{w_*}} \right) \overline{u_* w_*} V_0'(u_* w_*) \right)$$

by (6.1) and (6.8). Since we have

$$\begin{aligned}
\partial_t \left(\frac{1}{|w_*|^2} \right) &= -2\mathrm{Re}\, \left(\frac{\partial_t \overline{w_*}}{\overline{w_*}} \right) \cdot \frac{1}{|w_*|^2}, \\
\partial_t \left(\frac{|a_*|^2}{|w_*|^2} \right) &= \frac{n+2}{2|w_*|^2} \cdot \partial_t \left(|a_*|^2 \right), \\
|a_*|^2 \partial_t \left(\frac{1}{|w_*|^2} \right) &= \frac{n}{2|w_*|^2} \cdot \partial_t \left(|a_*|^2 \right)
\end{aligned}$$

by the definition of w_*, we obtain

$$2\mathrm{Re}\, I = 2\partial_t \left(\frac{|a_*|^2}{|w_*|^2} V_0(u_* w_*) \right) + \frac{n \partial_t (|a_*|^2)}{2|w_*|^2} \left(\overline{u_* w_*} V_0'(u_* w_*) - \frac{2(n+2)}{n} V_0(u_* w_*) \right). \tag{6.12}$$

So that, we have

$$\partial_t e_E^0 + \sum_{j=1}^n \partial_{x^j} e_E^j + e_E^{n+1} = 0$$

by (6.11) and (6.12). We obtain the required result by the integration for t and x. □

The example (6.4) satisfies the assumption (6.8), and we have

$$\overline{u_* w_*} V_0'(u_* w_*) - \frac{2(n+2)}{n} V_0(u_* w_*) = \frac{\lambda_0}{p+1} \left(p - 1 - \frac{4}{n} \right) |u_* w_*|^{p+1} \tag{6.13}$$

in the definition of e_E^{n+1} in Proposition 6.2, where we note that $p = 1 + 4/n$ is known as the pseudo-conformal number. Let us focus on the following two cases (i) and (ii).

(i) Let us consider the case $\theta = \omega^0 = \omega^1 = 0$ and $C_0 > 0$ (namely, $m/\hbar > 0$) in Proposition 6.2. Since we have $e^{2i\omega^0}/e^{2i(\theta+\omega^1)} = 1$, the equation (6.6) becomes the nonlinear Schrödinger equation

$$\pm i\frac{2m}{\hbar}\partial_t u_* + \frac{1}{|a_*|^2}\Delta_x u_* - \frac{1}{w_*}V_0'(u_* w_*) = 0.$$

We have the strict charge conservation

$$\int_{\mathbb{R}^n} e_C^0(t,x)dx = \int_{\mathbb{R}^n} e_C^0(0,x)dx$$

by (6.9) and $e_C^{n+1} = 0$. It is remarkable to see that the spatial variance does not affect the conservation of the charge. In other words, the total probability $\int_{\mathbb{R}^n} e_C^0(\cdot,x)dx$ does not change even if the space is expanding or contracting. When the space is invariant, namely, when $a_*(\cdot)$ is a constant, we have $e_E^{n+1} = 0$ and the strict energy conservation

$$\int_{\mathbb{R}^n} e_E^0(t,x)dx = \int_{\mathbb{R}^n} e_E^0(0,x)dx$$

by (6.10). When the space is expanding ($\partial_t a_*(\cdot) > 0$) or contracting ($\partial_t a_*(\cdot) < 0$), we have the dissipative property ($e_E^{n+1} > 0$) or the anti-dissipative property ($e_E^{n+1} < 0$) depending on the structure of V_0 and V_0'. For example, let us consider the example (6.4). By (6.13), we have the dissipative property $e_E^{n+1} > 0$ when

$$\partial_t a_*(\cdot)\lambda_0\left(p - 1 - \frac{4}{n}\right) > 0, \tag{6.14}$$

while we have the anti-dissipative property $e_E^{n+1} < 0$ when

$$\partial_t a_*(\cdot)\lambda_0\left(p - 1 - \frac{4}{n}\right) < 0.$$

We note that we do not have the dissipative and anti-dissipative properties, namely, $e_E^{n+1} = 0$, for the conformal power $p = 1 + 4/n$ even if the space is expanding or contracting.

(ii) Let us consider the case $\theta = \omega^0 = 0$, $\omega^1 = \pi/4$ and $C_0 > 0$ in Proposition 6.2. We have $e^{2i\omega^0}/e^{2i(\theta+\omega^1)} = -i$. The equation (6.6) with the positive sign is the parabolic equation

$$\frac{2m}{\hbar}\partial_t u_* - \frac{1}{|a_*|^2}\Delta_x u_* + \frac{1}{w_*}V_0'(u_* w_*) = 0.$$

Even if the space is invariant, namely, $a_*(\cdot)$ is a constant, we have the dissipative properties ($e_C^{n+1} > 0$) when $\overline{u_* w_*}V_0'(u_* w_*) \geq 0$. When the space is variant, the dissipative property ($e_E^{n+1} > 0$) and the anti-dissipative property ($e_E^{n+1} < 0$) depend on the spatial expansion ($\partial_t a_* > 0$), the spatial contraction ($\partial_t a_* < 0$) and the structure of V_0 and V_0'. For example, let us consider the example (6.4). By (6.13), we have the dissipative property $e_E^{n+1} > 0$ for the case (6.14). Moreover, we also have the dissipative property $e_E^{n+1} > 0$ even if

$$\partial_t a_*(\cdot)\lambda_0\left(p - 1 - \frac{4}{n}\right) = 0$$

since e_E^{n+1} contains the positive term $2C_0|a_*|^2|\partial_t u_*|^2$.

Therefore, by the above arguments in (i) and (ii), the dissipative and anti-dissipative properties of the equations (1.4) and (1.5) are dependent on the local coordinates (1.3), the scale-function $a(\cdot)$ and the structure of the nonlinear terms.

7. Appendix: Remarks on Vilenkin's model

Let us consider the model of the birth of the universe by Vilenkin and extend it on $\mathbb{M}^{1+n}$. We have derived the line element (3.11), which is from (3.1) with (3.2), (3.3) and (3.10). We put $\widetilde{f}(z) := h(z^0) + f(r)$ for the functions h and f in (3.1). By direct calculations, we have $g = -c^2 e^{n\widetilde{f}}$, $\partial_\alpha(-g)^{1/2} = n(-g)^{1/2}\partial_\alpha \widetilde{f}/2$ and

$$R = -\frac{n}{c^2}\partial_0^2 h - \frac{n(n+1)}{4c^2}(\partial_0 h)^2 - \frac{n(n-1)k^2}{q^2}e^{-h}.$$

By the integration by parts and $\partial_0(-g)^{1/2} = n(-g)^{1/2}\partial_0 h/2$, we have

$$\int_{\mathbb{M}^{1+n}} \partial_0^2 h(-g)^{1/2} dz = -\int_{\mathbb{M}^{1+n}} \partial_0 h \partial_0(-g)^{1/2} dz = -\frac{n}{2}\int_{\mathbb{M}^{1+n}} (\partial_0 h)^2 (-g)^{1/2} dz.$$

By these results, we have

$$\int_{\mathbb{M}^{1+n}} R(-g)^{1/2} dz = \frac{n-1}{2}\int_{\mathbb{M}^{1+n}} \left\{\frac{n}{2c^2}(\partial_0 h)^2 - \frac{2nk^2}{q^2}e^{-h}\right\}(-g)^{1/2} dz.$$

So that, we have

$$\int_{\mathbb{M}^{1+n}} (R + 2\Lambda)(-g)^{1/2} dz = \frac{(n-1)\kappa c^3}{2}\int_{\mathbb{M}^{1+n}} L(a, \partial_0 a)e^{nf/2} dz,$$

where we have defined $L(a, \partial_0 a)$ by

$$L(a, \partial_0 a) := \frac{a^n}{\kappa c^2}\left\{\frac{2n}{c^2}\left(\frac{\partial_0 a}{a}\right)^2 - \frac{2nk^2}{a^2 q^2} + \frac{4\Lambda}{n-1}\right\}.$$

We regard $L(a, \partial_0 a)$ as the Lagrangian for the variation of $a(\cdot)$. We define the momentum $p := \partial L/\partial(\partial_0 a)$ and the Hamiltonian $H := p\partial_0 a - L$. By the definition of $L(a, \partial_0 a)$, we have

$$p = \frac{4na^{n-2}\partial_0 a}{\kappa c^4}, \quad H = \frac{2na^n}{\kappa c^2}\left\{\frac{1}{c^2}\cdot\left(\frac{\partial_0 a}{a}\right)^2 + \frac{k^2}{q^2 a^2} - \frac{2\Lambda}{n(n-1)}\right\},$$

by which we also have

$$H = \frac{2na^n}{\kappa c^2}\left(\frac{c}{4na^{n-1}}\right)^2\left\{\kappa^2 p^2 c^4 + \frac{V(a)}{c^4}\right\},$$

where we have put a potential

$$V(a) := c^2(4na^{n-2})^2\left(\frac{k^2}{q^2} - \frac{a^2}{\ell^2}\right)$$

and $\ell := (n(n-1)/2\Lambda)^{1/2}$. So that, the solution of the equation of motion $H = 0$ is given by the scale-function

$$a(z^0) = \begin{cases} \pm\frac{k\ell}{q}\cosh\left(\frac{cz^0}{\ell} + C\right) & \text{if } k \neq 0, \\ a(0)e^{\pm cz^0/\ell} & \text{if } k = 0 \end{cases} \tag{7.1}$$

for some constant $C \in \mathbb{C}$. This is a natural extension of Vilenkin's model on $\mathbb{M}^{1+n}$. Let us consider the case $\pm k/q = 1$, $C = 0$, $\ell > 0$, $\kappa \in \mathbb{R}$, $\Lambda(\neq 0) \in \mathbb{R}$ and $z^0 = t \in \mathbb{R}$. Then we have $a(t) = \ell \cosh(ct/\ell)$. And $a(\cdot)$, L, p, H are real-valued. We need $a(\cdot) \geq \ell$ for the equation $H = 0$ since $V > 0$ and $H > 0$ if $a(\cdot) < \ell$. This means that the universe grows up as the de Sitter spacetime $a(t) = \ell \cosh(ct/\ell)$ for $t \geq 0$ in real time $z^0 = t$. To consider the excluded case $a(\cdot) < \ell$, let us use the imaginary time $z^0 = it$ for $t < 0$. Then we have $a(t) = \ell \cos(ct/\ell)$ for $t < 0$. We need $-\pi\ell/2c < t$ for $a(t) > 0$. Since $V(a) > 0$ for $-\pi\ell/2c < t < 0$, we obtain the model that the universe passes through the mountain of the potential $V(a) > 0$ by the tunnel effect in imaginary time $z^0 = it$ for $-\pi\ell/2c < t < 0$, then it grows up as the de Sitter spacetime in real time $z^0 = t$ for $t \geq 0$. On the other hand, the solution $a(z^0) = a(0)e^{\pm cz^0/\ell}$ in (7.1) for $k = 0$ and real time $z^0 = t$ does not need the tunnel effect.

8. Appendix: Remarks on the geodesic curves

Let us consider the geodesic curves and their Hamiltonians derived from the complex line element (2.6). We consider the generalized line element of (3.1) given by

$$-c^2(d\tau)^2 = -c^2(dz^0)^2 + g_{jk}dz^j dz^k$$

for arbitrary complex-valued functions $\{g_{jk}\}_{1\leq j,k\leq n}$ which satisfy the symmetry conditions $g_{jk} = g_{kj}$ for $1 \leq j, k \leq n$. We consider the non-relativistic velocity and the relativistic velocity, respectively.

(1) Let us consider the non-relativistic velocity $v^j := dz^j/dz^0$ for $1 \leq j \leq n$. Let (g^{jk}) be the inverse matrix of (g_{jk}). The change of upper and lower indices is done by g_{jk} and g^{jk}. We put $J := 1 - v^j v_j/c^2$, and we consider an arbitrary potential $U = U(z^0, \cdots, z^n)$. We denote the mass by m. We define the Lagrangian L, the momenta $\{p_j\}_{j=1}^n$, and the Hamiltonian H by

$$L := -mc^2 J^{1/2} - U, \quad p_j := \frac{\partial L}{\partial v^j}, \quad H := v^j p_j - L. \tag{8.1}$$

We put $K := m^2c^2 + p^j p_j$. By direct calculations, we have $d\tau = J^{1/2}dz^0$, $\partial J/\partial v^j = -2v_j/c^2$, $\partial J/\partial g_{jk} = -v^j v^k/c^2$, $p^j = mv^j J^{-1/2}$, $\partial L/\partial z^\alpha = -\partial U/\partial z^\alpha$, $\partial L/\partial g_{jk} = J^{1/2}p^j p^k/2m$. So that, the Euler-Lagrange equation for L is given by

$$\begin{aligned} 0 &= \frac{\partial L}{\partial z^j} - \frac{d}{dz^0}\frac{\partial L}{\partial v^j} + \frac{\partial L}{\partial g_{\ell m}}\frac{\partial g_{\ell m}}{\partial z^j} \\ &= -\frac{\partial U}{\partial z^j} - \frac{dg_{jk}}{dz^0}p^k - g_{jk}\frac{dp^k}{dz^0} + \frac{J^{1/2}}{2m}p^\ell p^m \frac{\partial g_{\ell m}}{\partial z^j}. \end{aligned} \tag{8.2}$$

We have

$$K^{1/2} = mcJ^{-1/2}, \quad H = cK^{1/2} + U \tag{8.3}$$

by $p^j p_j = m^2c^2(J^{-1} - 1)$. Since we have $\partial K/\partial p^j = 2p_j$ and $\partial K/\partial g_{jk} = p^j p^k$, we have $\partial H/\partial p^j = v_j$, $\partial H/\partial g_{jk} = cp^j p^k/2K^{1/2}$ and $\partial H/\partial z^\alpha = \partial U/\partial z^\alpha$. Therefore, we have

$$\frac{dH}{dz^0} = \frac{\partial U}{\partial z^0} - \frac{c}{2K^{1/2}}p^j \frac{\partial g_{jk}}{\partial z^0}p^k + \frac{c^2}{2K}p^j p^\ell p^m \frac{\partial g_{\ell m}}{\partial z^j} =: -H_R$$

by (8.2), where we have put the right hand side as $-H_R$. So that, the Hamiltonian H satisfies the conservation

$$H(z^0) + \int_0^{z^0} H_R(w)dw = H(0). \tag{8.4}$$

Now, we consider the transform (1.3) with $\omega^1 = \cdots = \omega^n$. We consider the case $(g_{jk}) := a(z^0)^2 \mathrm{diag}(1, \cdots, 1)$, namely, the case $q = 1$ and $k = 0$ in (3.11). Then we have

$$J = 1 - e^{2i(\omega^1 - \omega^0)} \frac{a(e^{i\omega^0} x^0)^2}{c^2} \sum_{j=1}^{n} \left(\frac{dx^j}{dx^0} \right)^2$$

by the definition of J. We also have

$$e^{i\omega^0} H_R = e^{2i(\omega^1 - \omega^0)} \frac{ma}{J^{1/2}} \frac{da}{dx^0} \sum_{j=1}^{n} \left(\frac{dx^j}{dx^0} \right)^2 - \frac{\partial U}{\partial x^0}$$

by $p^j = me^{i(\omega^1 - \omega^0)} J^{-1/2} dx^j/dx^0$, (8.3) and $\partial g_{\ell m}/\partial z^j = 0$. Especially, let us consider the case $m \geq 0$, $\omega^0 = 0$, $\omega^1 = 0, \pm\pi/2, \pi$. Let U and $a(\cdot)(> 0)$ be real-valued functions. Let us consider the small velocity such that $J > 0$. Then K, H, H_R are real-valued functions. Moreover, if $-\partial U/\partial x^0 \geq 0$ and

$$\frac{da}{dx^0} \begin{cases} \geq 0 & \text{for } \omega^1 = 0, \pi, \\ \leq 0 & \text{for } \omega^1 = \pm\frac{\pi}{2}, \end{cases}$$

then $H_R \geq 0$. So that, the spatial variance $da/dx^0 \neq 0$ has a dissipative effect on the conservation (8.4), while the spatial invariance $da/dx^0 = 0$ yields the strict conservation $H(x^0) = H(0)$ for $x^0 \in \mathbb{R}$ when the potential U is stationary, namely, $\partial U/\partial x^0 = 0$. Therefore, the conservation of the Hamiltonian depends on the spatial variance $a(\cdot)$.

(2) Let us consider the above argument for the relativistic velocity $v^\alpha := dz^\alpha/d\tau$ for $0 \leq \alpha \leq n$ for the general line element (2.6). We put $J := -v^\alpha g_{\alpha\beta} v^\beta$, and the potential $U = U(z^0, \cdots, z^n)$. We denote the mass by m. We define the Lagrangian L, the momenta $\{p_\alpha\}_{\alpha=0}^n$, and the Hamiltonian H by

$$L := -mcJ^{1/2} - U, \quad p_\alpha := \frac{\partial L}{\partial v^\alpha}, \quad H := v^\alpha p_\alpha - L. \tag{8.5}$$

We have

$$\frac{\partial L}{\partial z^\alpha} = -\frac{\partial U}{\partial z^\alpha}, \quad p_\alpha = -\frac{mc}{2J^{1/2}} \cdot \frac{\partial J}{\partial v^\alpha}, \quad \frac{\partial L}{\partial g_{\alpha\beta}} = -\frac{mc}{2J^{1/2}} \cdot \frac{\partial J}{\partial g_{\alpha\beta}}.$$

The Euler-Lagrange equation for L is given by

$$\frac{\partial L}{\partial z^\gamma} - \frac{d}{d\tau}\left(\frac{\partial L}{\partial v^\gamma} \right) + \frac{\partial L}{\partial g_{\alpha\beta}} \cdot \frac{\partial g_{\alpha\beta}}{\partial z^\gamma} = 0. \tag{8.6}$$

Since $\partial J/\partial v^\gamma = -2v_\gamma$ and $\partial J/\partial g_{\alpha\beta} = -v^\alpha v^\beta$, we have $v_\alpha = J^{1/2} p_\alpha/mc$ and $\partial L/\partial g_{\alpha\beta} = mcv^\alpha v^\beta/2J^{1/2}$. The equation (8.6) is rewritten as

$$-\frac{\partial U}{\partial z^\gamma} - \frac{d}{d\tau}\left(\frac{mc}{J^{1/2}} \right) \cdot v_\gamma - \frac{mc}{J^{1/2}} \cdot \frac{dg_{\gamma\delta}}{d\tau} \cdot v^\delta - \frac{mc}{J^{1/2}} \cdot g_{\gamma\delta} \cdot \frac{dv^\delta}{d\tau} + \frac{mc}{2J^{1/2}} \cdot \frac{\partial g_{\alpha\beta}}{\partial z^\gamma} \cdot v^\alpha v^\beta = 0. \tag{8.7}$$

Multiplying v^γ to the both sides in (8.7) and using the elementary facts

$$\frac{dU}{d\tau} = v^\gamma \frac{\partial U}{\partial z^\gamma}, \quad J = -v^\gamma v_\gamma,$$

$$v^\gamma g_{\gamma\delta} \frac{dv^\delta}{d\tau} = \frac{1}{2}\left(\frac{d}{d\tau}\left(v^\gamma v_\gamma\right) - v^\gamma \frac{dg_{\gamma\delta}}{d\tau} v^\delta\right), \quad \frac{\partial g_{\alpha\beta}}{\partial z^\gamma} v^\gamma = \frac{dg_{\alpha\beta}}{d\tau},$$

we have

$$\frac{dU}{d\tau} = 0.$$

Since we have $H = U$ by $v^\gamma v_\gamma = -J$ and the definitions of H and L, the Hamiltonian H is a constant function independent of the proper time τ. Namely, we have

$$H(\tau) = H(0). \tag{8.8}$$

Comparing the conservation laws (8.4) and (8.8), we know that the Hamiltonian H is strictly conserved with respect to the proper time τ independently of the scale-function $a(\cdot)$, while the Hamiltonian is dependent on the scale-function with respect to the local time z^0.

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